SIF8015 Logic

Solutions to exercise 6
Predicate Calculus
Due March 16.

Task 1

Let $\mathcal{L}$ denote a first order language, I an interpretation and v a valuation in I. Let $\text{Wf}_{\mathcal{L}}$ denote the set of wfs. of $\mathcal{L}$. Draw a diagram that shows the relations between $\text{Wf}_{\mathcal{L}}$ and the following subsets of $\text{Wf}_{\mathcal{L}}$:

$\text{Taut}_{\mathcal{L}}$	$\{\mathcal{A} \mid \mathcal{A}$ is a tautology of $\mathcal{L}\}$
$\text{Th}_{\mathcal{L}}$	$\{\mathcal{A} \mid \mathcal{A}$ is a theorem of $K_{\mathcal{L}}\}$
$\text{LV}_{\mathcal{L}}$	$\{\mathcal{A} \mid \mathcal{A}$ is logically valid$\}$
$\text{Contr}_{\mathcal{L}}$	$\{\mathcal{A} \mid \mathcal{A}$ is contradictory$\}$
$\text{True}_{\mathcal{L},I}$	$\{\mathcal{A} \mid \mathcal{A}$ is true in $I\}$
$\text{False}_{\mathcal{L},I}$	$\{\mathcal{A} \mid \mathcal{A}$ is false in $I\}$
$\text{Sat}_{\mathcal{L},I,v}$	$\{\mathcal{A} \mid \mathcal{A}$ is satisfied by v in $I\}$
$\text{Unsat}_{\mathcal{L},I,v}$	$\{\mathcal{A} \mid \mathcal{A}$ is not satisfied by v in $I\}$

You should carefully explain your answers, for instance, give the name of some result in the textbook that justifies each of the relations in the diagram.

A diagram of the relations can be seen in Figure 1. Please note the following:

1) The universe of wfs. is divided into those formulae which are satisfied in v and those which are not satisfied in v. Some of these formulae are neither true nor false in a given interpretation. Consider for example the formula $p(x)$ in the interpretation, I, with $D = \{a, b\}$ and valuation, v, where $v(x) = a$, $v(p(a)) = T$ and $v(p(b)) = F$. This formula is satisfied by v in I, but it is not true in I.

2) There are formulae which are true in a given interpretation, but which are not logically valid. Consider for example the mathematical expression $(\forall x)(\forall y)(x + y) \geq 0$. This is true if $D =$ "all natural numbers", but it is not true if $D =$ "all integers".

3) The set of logically valid formulae and the set of theorems are the same sets (The Soundness Theorem (Prop 4.5) and the Adequacy Theorem (Prop. 4.41)).

4) There are logically valid formulae which are not tautology's. Consider for example the following valid formula: $(\forall x)(A_1^1(x) \rightarrow (A_1^1(x))$. This formula can also be written as $(\forall x)\mathcal{A}$, which is clearly not a tautology in general.

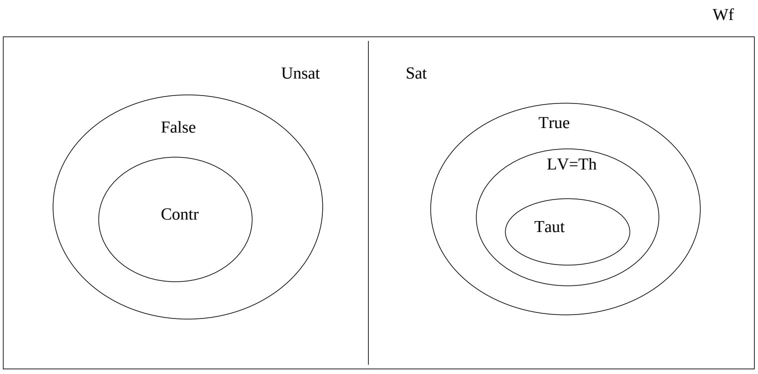

Figure 1: Diagram

Task 2

In this task we will take a closer look at the integers modulo 5, $\mathbb{Z}_5$. In order to do so we will define a first order language that is appropriate for making statements about $\mathbb{Z}_5$. Let $\mathcal{L}$ denote the first order language with the following symbols, in addition to the usual variables, punctuation symbols, connectives, and the universal quantifier:

- Individual constants a_1, a_2, a_3, a_4, a_5
- Function letters $f_1^1, f_1^2, f_2^2, f_3^2$
- Predicate letters $A_1^1, A_2^1, A_3^1, A_1^2, A_2^2, A_3^2$

In order to give our symbols meaning we define the following interpretation I: The domain is given by $D_I = \{\bar{a}_1, \ldots, \bar{a}_5\} = \{0, 1, 2, 3, 4\}$.

The interpretation of the individual constants, function letters, and predicate letters are as follows:

Constants:

a_1	stands for	$\bar{a_1}$ (= 0)
a_2	stands for	$\bar{a_2}$ (= 1)
a_3	stands for	$\bar{a_3}$ (= 2)
a_4	stands for	$\bar{a_4}$ (= 3)
a_5	stands for	$\bar{a_5}$ (= 4)

Functions:

f_1^1	stands for	$\bar{f}_1^1(\bar{a}_j) = (\bar{a}_j + 1) \bmod 5$	Successor function
f_1^2	stands for	$\bar{f}_1^2(\bar{a}_i, \bar{a}_j) = (\bar{a}_i + \bar{a}_j) \bmod 5$	Addition
f_2^2	stands for	$\bar{f}_2^2(\bar{a}_i, \bar{a}_j) = (\bar{a}_i * \bar{a}_j) \bmod 5$	Multiplication
f_3^2	stands for	$\bar{f}_3^2(\bar{a}_i, \bar{a}_j) = (\bar{a}_i - \bar{a}_j) \bmod 5$	Subtraction

Predicates:

A_1^1	stands for	'=0'	i.e., $A_1^1(x)$ means x is 0
A_2^1	stands for	'is even'	i.e., $A_2^1(x)$ means x is an even number
A_3^1	stands for	'is odd'	i.e., $A_3^1(x)$ means x is an odd number
A_1^2	stands for	'='	i.e., $A_1^2(x, y)$ means x equals y
A_2^2	stands for	'>'	i.e., $A_2^2(x, y)$ means y is greater than x
A_3^2	stands for	'divides'	i.e., $A_3^2(x, y)$ means x divides y

a) The language $\mathcal{L}$ is finite. Is the number of terms finite? Explain.

The number of terms in the language $\mathcal{L}$ is infinite. To understand why, remember the recursive definition of a term (Definition 3.6).

b) Write the complete tables specifying the functions $\bar{f}_1^2$, $\bar{f}_2^2$, and $\bar{f}_3^2$, i.e., the functions that are the interpretations of the function letters f_1^2, f_2^2, and f_3^2. For $\bar{f}_1^1$ the table look like:

x	$\bar{f}_1^1(x)$
0	1
1	2
2	3
3	4
4	0

You may use the numbers 0, ..., 4 as abbreviations for $\bar{a}_1$, ..., $\bar{a}_5$. Hint: For functions with two arguments you should end up with a 5×5 table.

Function - tables:

$\bar{f}_1^2(x, y)$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\bar{f}_2^2(x, y)$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$\bar{f}_3^2(x, y)$	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

c) Write the complete tables specifying the relations $\bar{A}_2^1$, $\bar{A}_3^1$, $\bar{A}_1^2$, $\bar{A}_2^2$, and $\bar{A}_3^2$, i.e., the interpretations of the predicate letters A_2^1, A_3^1, A_1^2, A_2^2, and A_3^2. For $\bar{A}_1^1$ the table look like:

x	$\bar{A}_1^1(x)$
0	T
1	F
2	F
3	F
4	F

Use T or F as appropriate in the given cell. Hint: For relations with two arguments the table should have 5 rows and 5 columns.

Predicate - tables:

x	$\bar{A}_2^1(x)$
0	T
1	F
2	T
3	F
4	T

x	$\bar{A}_3^1(x)$
0	F
1	T
2	F
3	T
4	F

$\bar{A}_1^2(x,y)$	0	1	2	3	4
0	T	F	F	F	F
1	F	T	F	F	F
2	F	F	T	F	F
3	F	F	F	T	F
4	F	F	F	F	T

$\bar{A}_2^2(x,y)$	0	1	2	3	4
0	F	T	T	T	T
1	F	F	T	T	T
2	F	F	F	T	T
3	F	F	F	F	T
4	F	F	F	F	F

$\bar{A}_3^2(x,y)$	0	1	2	3	4
0	T	F	F	F	F
1	T	T	T	T	T
2	T	F	T	F	T
3	T	F	F	T	F
4	T	F	F	F	T

d) Let v be some unspecified but fixed valuation in I. Evaluate the following terms, i.e., find $v(t)$ for all the given ts. For example:

$$v(f_1^1(a_1)) = \bar{f}_1^1(v(a_1)) = \bar{f}_1^1(\bar{a}_1)) = \bar{f}_1^1(0) = 1$$

i) $f_1^2(f_1^1(f_1^1(f_1^1(a_1))), a_3)$

ii) $f_2^2(a_4, f_3^2(a_4, a_2))$

iii) $f_1^2(a_4, a_5)$

Would this be different for another valuation.

i) $v(f_1^2(f_1^1(f_1^1(f_1^1(a_1))), a_3)) = 0$

ii) $v(f_2^2(a_4, f_3^2(a_4, a_2))) = 1$

iii) $v(f_1^2(a_4, a_5)) = 2$

This would not be any different for another valuation, because the terms contain constants only and a constant always evaluate to the same value.

e) Let v be any valuation in I. Which of the following wfs. of $\mathcal{L}$ are satisfied by v?

i) $A_1^1(f_3^2(f_1^1(f_1^1(f_1^1(a_1))), a_4))$
ii) $A_2^1(f_2^2(a_4, f_3^2(a_4, a_2)))$
iii) $A_3^1(f_1^2(a_4, a_5))$

What do they express? Which are true in I?

Corresponding mathematical expressions:

i) $3 - 3 = 0$
ii) $3 * (3 - 1)$ mod 5 is even
iii) $3 + 4$ mod 5 is odd

i) is satisfied by v. Since the formulae only contain constants, i) is also true in I.

f) Assume that we restrict $\mathcal{L}$ to having only the five variables $x_1, \ldots, x_5$. How many distinct valuations are there? Let v be any valuation in I. How many valuations are 1-equivalent, 2-equivalent, and 5-equivalent to v? Explain your answers.

There are 5^5 distinct valuations in the restricted language $\mathcal{L}$ with only five variables. There are 5^3 valuations which are 1-equivalent, 2-equivalent, and 5-equivalent to v.

g) Let v be the valuation where $v(x_1) = \bar{a_5}, v(x_2) = \bar{a_3}, v(x_3) = \bar{a_1}, v(x_4) = \bar{a_2}, v(x_5) = \bar{a_5}$ Which of the following wfs. are satisfied in v in our interpretation? Which of the following wfs. are true in our interpretation?

i) $(\sim A_1^2(a_1, f_1^1(x_1)))$
ii) $A_1^2(f_1^1(x_2), f_1^1(x_3)) \to A_1^2(x_2, x_3)$
iii) $A_1^2(f_1^2(x_4, f_1^1(x_5)), f_1^1(f_1^2(x_4, x_5)))$
iv) $(\forall x_1)A_1^2(f_2^2(x_1, a_1), a_1)$
v) $(\forall x_1)(\forall x_2)A_1^2(f_2^2(x_1, f_1^1(x_2)), f_1^2(f_2^2(x_1, x_2), x_1))$

ii), iii), iv) and v) are all satisfied by v in our interpretation. ii), iii), iv) and v) are also true in our interpretation.

h) How would you express $(\forall x_1)A_1^1(x_1)$ in statement calculus? Do you think it is possible to express all our knowledge about $\mathbb{Z}_5$ in statement calculus? Do you think this would be different if the domain was infinite? Explain your answers.

In statement calculus the formula $(\forall x_1)A_1^1(x_1)$ could be expressed like $(A \wedge B \wedge C \wedge D \wedge E)$, where A is interpreted as $\bar{a_1} = 0$, B as $\bar{b_1} = 0$, ... In this manner it is possible to express all our knowledge about $\mathbb{Z}_5$ in statement calculus. However, this would not be possible if the domain was infinite, because then the formula $(\forall x_1)A_1^1(x_1)$ would contain an infinite number of conjuncts.

i) Let K' be the formal system $K_{\mathcal{L}}$, with K1, ..., K6 as the usual axioms, and the usual inference rules MP and GEN (Generalisation). Show that the axioms K4, K5 and K6 are not logically valid if the restrictions are not satisfied. Use counterexamples from our interpretation and find instances of the axioms that are not true in the interpretation.

Counterexamples:

K4 When proving Proposition 4.4 A.G. Hamilton does not mention the restriction. As far as we know K4 is thus valid both with and without the restriction.

K5 Let $\mathcal{A}(x_1) = (\exists x_2)(\sim A_1^2(x_1, a_5) \to A_2^2(x_2, x_1))$ and $t = x_2$ in K5 (notice that t is not free for x_1). We now have a formula, $(\forall x_1)(\exists x_2)(\sim A_1^2(x_1, a_5) \to A_2^2(x_2, x_1)) \to (\exists x_2)(\sim A_1^2(x_2, a_5) \to A_2^2(x_2, x_2))$, which is not true in our interpretation!

K6 Consider the formula $(\forall x)(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to (\forall x)\mathcal{B})$ where $x \in f_v(\mathcal{A})$. Interpret $\mathcal{A}$ as $A_1^1(x)$ and $\mathcal{B}$ as $\bar{A}_1^1(x)$ such that $A_1^1 = \bar{A}_1^1 = \{0\}$ and $v(x) = 0$. Now $(\forall x)(\mathcal{A} \to \mathcal{B})$ is true and thus satisfied in v, while $(\mathcal{A} \to (\forall x)\mathcal{B})$ is not satisfied by v. Thus $(\forall x)(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to (\forall x)\mathcal{B})$ is not true in this interpretation and K6 is thus not logically valid without the restriction.

j) Is K' sound? Explain why/why not.

K' is sound, since $K_{\mathcal{L}}$ is sound and K' is only a restricted formal system of $K_{\mathcal{L}}$.

k) Let

EQ1 :	$(\forall x_1)\ A_1^2(x_1, x_1)$	Reflexivity
EQ2 :	$(\forall x_1)(\forall x_2)\ (A_1^2(x_1, x_2) \to A_1^2(x_2, x_1))$	Symmetry
EQ3 :	$(\forall x_1)(\forall x_2)(\forall x_3)\ (A_1^2(x_1, x_2) \to (A_1^2(x_2, x_3) \to A_1^2(x_1, x_3)))$	Transitivity
EQ4 :	$(\forall x_1)(\exists x_2)\ A_1^2(x_1, x_2)$	Totality

Note that the interpretation of totality is that "for all x_1, there is a x_2 such that $x_1 = x_2$".

Let K'' be K' with EQ1, EQ2, EQ3, and EQ4 as additional axioms.

i) How would you go about to prove whether K'' is sound?

ii) Is K'' complete? Is K'' consistent? Explain your answers.

iii) Does there exist a consistent complete extension of K''? Explain your answer.

Note, that you are not required to write proofs for these questions, textual explanations are sufficient.

i) K' is sound since K1, ..., K6 are logically valid, and MP and GEN are sound inference rules. Thus to prove that K'' is sound we only have to prove that EQ1, ..., EQ4 are logically valid (Which, by the way, they are not!).

ii) K'' is not complete. To prove this it is sufficient to find a formula, $\mathcal{A}$, such that neither $\mathcal{A}$ nor $\sim \mathcal{A}$ is a theorem of K''. For example $A_1^1(x)$. From prop. 4.43 and 4.44 we know that a formal system is consistent if it exists an interpretation in which every axiom of this system is true. K' is consistent,and thus we only have to find an interpretation in which EQ1, ..., EQ4 also are true. For example the interpretation given in this exercises. Thus K'' is consistent.

iii) Since K'' is consistent there exists a consistent extension which is complete (Prop 4.39).

l) Write a definite logic program P that expresses the statements EQ2, EQ3, and EQ4. Show, by constructing an SLD-tree, that EQ1 is a consequence of P. Recall that if θ is the computed answer substitution for a goal $\leftarrow G$ via a computation rule $\mathcal{R}$, then $(G\theta)'$, the universal closure of $G\theta$, is a logical consequence of the program. Hints: 1) Try the goal $\leftarrow A_1^2(X, X)$. 2) Ignore all infinite branches, that is, it suffices to find a finite path that leads to a refutation.

A definite logic program P:

1) $r(x, y) \leftarrow r(y, x)$.
2) $r(x, y) \leftarrow r(x, z), r(z, y)$.
3) $r(x, h(x))$.

A SLD-tree for $r(x, x)$ can be seen in figure 2.

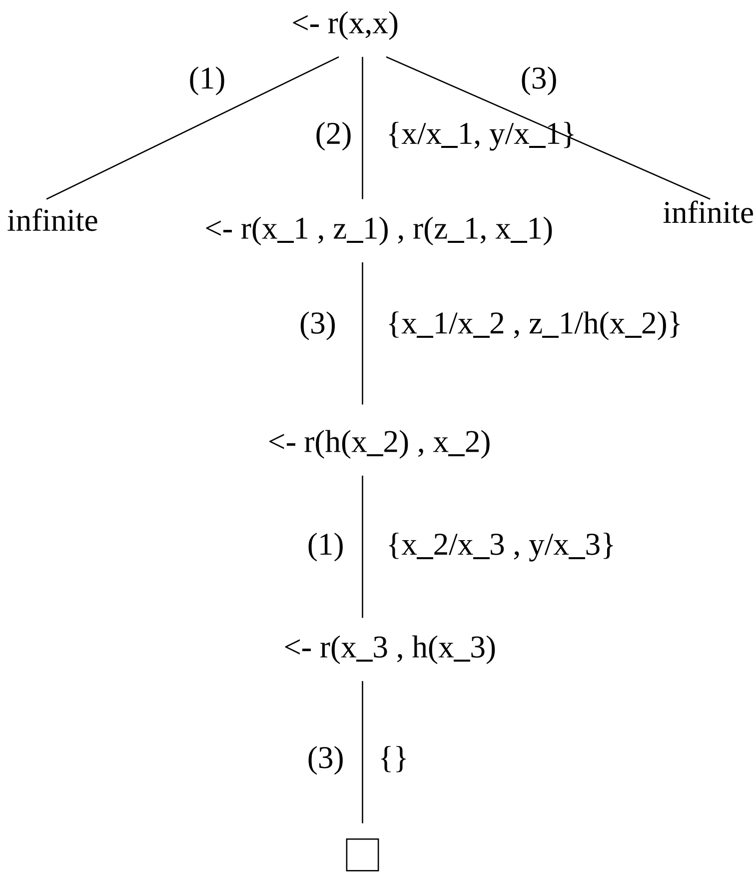

Figure 2: SLD-tree

Task 3

Consider the following 13 statements about faculty members.

1. Good-natured tenured mathematics professors are dynamic.
2. Grumpy student advisors play slot machines.
3. Smokers wearing Hawaiian shirts are phlegmatic.
4. Comical student advisors are mathematics professors.
5. Untenured faculty who smoke are nervous.
6. Phlegmatic tenured faculty members who wear Hawaiian shirts are comical.
7. Student advisors who are not stock market players are scholars.
8. Relaxed student advisors are creative.
9. Creative scholars who do not play slot machines wear Hawaiian shirts.
10. Nervous smokers play slot machines.
11. Student advisors who play slot machines are nonsmokers.
12. Creative stock market players who are good-natured wear Hawaiian shirts.
13. No student advisors are smokers.

Note the following:

Property	Opposite
good-natured	grumpy
tenured	untenured
dynamic	phlegmatic
relaxed	nervous

a) Formulate the 13 sentences above in predicate calculus. You may assume that the universe consists of people only. That is, you don't have to use a separate predicate to say "is person".

b) For each of the 13 wfs. in a), find a wf. in clausal form which is weakly equivalent to it.

c) If we consider a particular, but unspecified, faculty member a, the statement "a is good-natured" can be expressed by the proposition G_a. That is, if you have a predicate G that is interpreted such that, for a variable x, $G(x)$ means "x is good-natured", there is a straightforward translation into statement calculus of the formula $G(a)$ for a particular a. Using the same scheme for all predicates you defined in part a), formulate the 13 sentences above in statement calculus as statements about a particular a. You may want to use the clausal forms found in part b).

d) **Optional:** Prove in statement calculus that the last statement (13) is a consequence of the first 12. You may want to use resolution in statement calculus.

Resolution in statement calculus:

1. Transform each compound statement into clausal form.
2. Assume the negation of the clause that you are going to prove.
3. Arrive at a contradiction from the clauses from 1) and 2), using only the following inference rule:

   For any clauses $B \vee A_1 \vee \cdots \vee A_n$, and $\sim B \vee C_1 \vee \cdots \vee C_m$, where $B, A_i, C_j, i = 1, \ldots, n, j = 1, \ldots, m$ are literals (i.e., a statement variable or the negation of a statement variable)

   $$\frac{B \vee A_1 \vee \cdots \vee A_n \qquad \sim B \vee C_1 \vee \cdots \vee C_m}{A_1 \vee \cdots \vee A_n \vee C_1 \vee \cdots \vee C_m}$$

   Note that you may have to reorder the literals in some of the disjunctions in order to obtain clauses of the form given in this rule.

We will use the following predicates:

	Property	Opposite
$A(x)$	x is good-natured	grumpy
$B(x)$	x is tenured	untenured
$C(x)$	x is mathematics professor	
$D(x)$	x is dynamic	phlegmatic
$E(x)$	x wears Hawaiian shirts	
$F(x)$	x is a smoker	nonsmoker
$G(x)$	x is comical	
$H(x)$	x is relaxed	nervous
$I(x)$	x is a stock market player	
$J(x)$	x is scholar	
$K(x)$	x is creative	
$L(x)$	x plays slot machines	
$M(x)$	x is a student advisor	

a) and b) The sentences in predicate calculus and clausal from:

	Predicate calculus	Clausal form
1)	$(\forall x)((A(x) \wedge B(x) \wedge C(x)) \to D(x))$	$\sim A(x) \vee \sim B(x) \vee \sim C(x) \vee D(x)$
2)	$(\forall x)((\sim A(x) \wedge M(x)) \to L(x))$	$A(x) \vee \sim M(x) \vee L(x)$
3)	$(\forall x)((F(x) \wedge E(x)) \to \sim D(x))$	$\sim F(x) \vee \sim E(x) \vee \sim D(x)$
4)	$(\forall x)((G(x) \wedge M(x)) \to C(x))$	$\sim G(x) \vee \sim M(x) \vee C(X)$
5)	$(\forall x)((\sim B(x) \wedge F(x)) \to \sim H(x))$	$B(x) \vee \sim F(x) \vee \sim H(x)$
6)	$(\forall x)((\sim D(x) \wedge B(x) \wedge E(x)) \to G(x))$	$D(x) \vee \sim B(x) \vee \sim E(x) \vee G(x)$
7)	$(\forall x)((M(x) \wedge \sim I(x)) \to J(x))$	$\sim M(x) \vee I(x) \vee J(x)$
8)	$(\forall x)((H(x) \wedge M(x)) \to K(x))$	$\sim H(x) \vee \sim M(x) \vee K(x)$
9)	$(\forall x)((K(x) \wedge J(x) \wedge \sim L(x)) \to E(x))$	$\sim K(x) \vee \sim J(x) \vee L(x) \vee E(x)$
10)	$(\forall x)((\sim H(x) \wedge F(x)) \to L(x))$	$H(x) \vee \sim F(x) \vee L(x)$
11)	$(\forall x)((M(x) \wedge L(x)) \to \sim F(x))$	$\sim M(x) \vee \sim L(x) \vee \sim F(x)$
12)	$(\forall x)((K(x) \wedge I(x) \wedge A(x)) \to E(x))$	$\sim K(x) \vee \sim I(x) \vee \sim A(x) \vee E(x)$
13)	$(\forall x)(M(x) \to \sim F(x))$	$\sim M(x) \vee \sim F(x)$

c) We define symbols in statement calculus such that A means $A(x)$, that is, x is good-natured. Similarly for B, C, ... The sentences in statement calculus:

	Statement calculus (in clausal from)
1)	$\sim A \vee \sim B \vee \sim C \vee D$
2)	$A \vee \sim M \vee L$
3)	$\sim F \vee \sim E \vee \sim D$
4)	$\sim G \vee \sim M \vee C$
5)	$B \vee \sim F \vee \sim H$
6)	$D \vee \sim B \vee \sim E \vee G$
7)	$\sim M \vee I \vee J$
8)	$\sim H \vee \sim M \vee K$
9)	$\sim K \vee \sim J \vee L \vee E$
10)	$H \vee \sim F \vee L$
11)	$\sim M \vee \sim L \vee \sim F$
12)	$\sim K \vee \sim I \vee \sim A \vee E$
13)	$\sim M \vee \sim F$

d) We will use resolution in statement calculus to prove $\sim M \vee \sim F$. Note that 13′) and 14) are the negation of this claus. Note also that we are using commas instead of $\vee$.

	Using	Gives		Using	Gives
13′)		M	28)	1,27	$\sim B, \sim C, D$
14)		F	29)	26,28	$\sim C, D$
15)	2,13′	A, L	30)	17,29	$\sim G, D$
16)	3,14	$\sim E, \sim D$	31)	6,30	$D, \sim B, \sim E$
17)	4,13′	$\sim G, C$	32)	26,31	$D, \sim E$
18)	5,14	$B, \sim H$	33)	16,32	$\sim E$
19)	7,13′	I, J	34)	9,33	$\sim K, \sim J, L$
20)	8,13′	$\sim H, K$	35)	25,34	$\sim J, L$
21)	10,14	H, L	36)	23,35	$\sim J$
22)	11,13′	$\sim L, \sim F$	37)	12,19	$\sim K, \sim A, E, J$
23)	14,22	$\sim L$	38)	36,37	$\sim K, \sim A, E$
24)	21,23	H	39)	25,38	$\sim A, E$
25)	20,24	K	40)	27,39	E
26)	18,24	B	41)	33,40	$\{\}$
27)	15,23	A			