SIF8015 Logic

Exercise 4, Solutions
Informal Predicate Calculus

Task 1

Formulate the following mathematical concepts in predicate logic (You are free to use any predicate names you like, which means you do not have to use the A^i_j - notation from the book):

a) Let $\{a_n\}_{n\in\mathbb{N}}$ be a series of real numbers. The series converges if there exists a real number a, which is its limit.

b) The function f is continuous in point x.

c) Any function that is derivable in any point is continuous in any point.

a) Let the predicate letters S, R ,L and C be interpreted as:

S	'is a series of real numbers'
R	'is a real number'
L	'is the limit of the series'
C	'converges'

$$((\forall x_1)(\exists x_2)((S(x_1) \wedge R(x_2) \wedge L(x_2, x_1)) \rightarrow C(x_1)))$$

b) Let the predicate letters F, P and C be interpreted as:

F	'is a function'
P	'is a point'
C	'is continuous at'

$$((\exists x_1)(\exists x_2)(F(x_1) \wedge P(x_2) \wedge C(x_1, x_2)))$$

c) Let the predicate letters F, P, D and C be interpreted as:

F	'is a function'
P	'is a point'
D	'is derivable at'
C	'is continuous at'

$$(\forall x_1)(\forall x_2)((F(x_1) \wedge P(x_2) \wedge D(x_1, x_2)) \rightarrow C(x_1, x_2))$$

Alternatively; $(\forall x_1)(\forall x_2)((F(x_1) \wedge P(x_2) \wedge D(x_1, x_2)) \rightarrow (\forall x_3)(P(x_3) \rightarrow C(x_1, x_2)))$

Task 2

Which of the following strings of symbols are wfs.? If they are not, explain why!

a) $A_1^2(f_1^1(x_1), x_1)$

b) $f_1^3(x_1, x_3, x_4)$

c) $(A_1^1(x_2) \rightarrow A_1^3(x_3, a_1))$

d) $\sim (\forall x_2)A_1^2(x_1, x_2)$

e) $((\forall x_2)A_1^1(x_1) \rightarrow (\sim A_1^1(x_2)))$

f) $A_1^3(f_2^3(x_1, x_2, x_3))$

g) $(\sim A_1^1(x_1) \rightarrow A_1^1(x_2))$

h) $(\forall x_1)A_1^3(a_1, a_2, f_1^1(a_3))$

a) wf.

b) This is a term, not a wf.

c) A_1^3 needs to have three parameters.

d) Lacks parentheses around '$\sim$ - expression'.

e) wf.

f) A_1^3 needs to have three parameters.

g) Lacks parentheses around '$\sim$ - expression'.

h) wf.

Task 3

For each of the following wfs.:

1. Point out all free and bound occurences of variables.
2. Decide if the term $t = f_1^2(x_1, x_2)$ is free for x_1 in the wf.

a) $A_1^2(x_1, x_2) \rightarrow (\forall x_2)A_1^1(x_2)$

b) $((\forall x_2)A_1^2(x_2, a_1)) \vee ((\exists x_2)A_1^2(x_1, x_2))$

c) $(\forall x_1)A_1^2(x_1, x_2)$

d) $(\forall x_2)A_1^2(x_1, x_2)$

e) $(\forall x_2)A_1^1(x_2) \to A_1^2(x_1, x_2)$

Only bound occurences are pointed out. All other occurences, except those belonging to a quantifier, are free.

a) The second occurrence of x_2 is bound. Term t is free for x_1.

b) Both occurrences of x_2 are bound. Term t is NOT free for x_1.

c) x_1 is bound. Term t is free for x_1.

d) x_2 is bound. Term t is NOT free for x_1.

e) The first occurrence of x_2 is bound. Term t is free for x_1.

Task 4

Do exercise 13 on page 59 in the textbook by Hamilton.

Let A_1^2 stand for $<$ and let $D_N = \{0, 1, 2, 3, ...\}$. Now, for no natural number does there exist a second natural number such that this number is both greater and smaller than the first number.

Task 5

Do exercises 14c, 14d, and 14e on page 69 in the textbook by Hamilton.

(14c) In the interpretation N the formula equals the mathematical expression

$$x_1 * x_2 \neq x_2 * x_3$$

v with $v(x_1) = 2$, $v(x_2) = 4$ and $v(x_3) = 8$ satisfies this formula.
v' with $v'(x_1) = 2$, $v'(x_2) = 2$ and $v'(x_3) = 2$ does NOT satisfy this formula.

(14d) In the interpretation N the formula equals the mathematical expression

$$\text{for all } x_1 \quad : \quad x_1 * x_2 = x_3$$

v with $v(x_2) = 0$ and $v(x_3) = 0$ satisfies this formula.
v' with $v'(x_2) = 4$ and $v'(x_3) = 2$ does NOT satisfy this formula.

(14e) In the interpretation N the formula equals the mathematical expression

$$(\text{for all } x_1 : x_1 * 0 = x_1) \Rightarrow (x_1 = x_2)$$

This formula is always true in the interpretation N.

Task 6

Do exercises 16c, 16d, 17c, and 17d on page 69 in the textbook by Hamilton. Clearly explain your answers!

(16c) TRUE. For all natural numbers x_1 and x_2 there will always exist a third natural number x_3 such that

$$x_1 + x_2 = x_3$$

(16d) TRUE. There exists a natural number x_1 such that

$$x_1 * x_1 = x_1 + x_1$$

For example, $v(x_1) = 2$.

(17c) TRUE. For all integers x_1, x_2 and x_3 the following relation holds:

$$(x_1 < x_2) \to (x_1 - x_3 < x_2 - x_3)$$

(17d) TRUE. For all integers x_1 there will always exist a second integer x_2 such that

$$x_1 < x_1 - 2x_2$$

For example, $v(x_2) = -12$.

Task 7

Do exercises 19c and 19d on page 69-70 in the textbook by Hamilton.

(19c) Let I be an interpretation and let v be a valuation in I satisfying $(\forall x_1)(\mathcal{A} \to \mathcal{B})$. Then every v', 1-equivalent to v, satisfies $(\mathcal{A} \to \mathcal{B})$.

Now suppose that v does not satisfy $(\forall x_1)\mathcal{A} \to (\forall x_1)\mathcal{B}$. Then v satisfies $(\forall x_1)\mathcal{A}$ and v does not satisfy $(\forall x_1)\mathcal{B}$, and so there is a v', 1-equivalent to v, which satisfies $\mathcal{A}$ and does not satisfy $\mathcal{B}$. This v' does not satisfy $\mathcal{A} \to \mathcal{B}$, which contradicts our assumption!

(19d) Let I be an interpretation and let v be a valuation in I satisfying $(\forall x_1)(\forall x_2)\mathcal{A}$. Then every v', 1-equivalent to v, satisfies $(\forall x_2)\mathcal{A}$. Then every v'', 2-equivalent to v', satisfies $\mathcal{A}$.

Now suppose that v does not satisfy $(\forall x_2)(\forall x_1)\mathcal{A}$. Then there is a w', 1-equivalent to v, that does not satisfy $(\forall x_1)\mathcal{A}$. Then there is a w'', 2-equivalent to v', which does not satisfy $\mathcal{A}$.

Now, all v'' that are 1-equivalent and 2-equivalent to v satisfy $\mathcal{A}$, but there is a w'', also 1-equivalent and 2-equivalent to v, which does not satisfy $\mathcal{A}$. Hence we have a contradiction!

Task 8

Which of the following statements are true. Explain briefly why or why not.

1. Let I be an interpretation, let $\mathcal{A}$ be a wf. having x_i as its only free variable, and let v and v' be two i-equivalent valuations in I. Then v satisfies $\mathcal{A}$ if and only if v' satisfies $\mathcal{A}$.

2. Every logically valid formula is closed.

3. If $\mathcal{A}$ is a wf. that is satisfied by any valuation v in any interpretation I, then $(\sim \mathcal{A})$ is contradictory.

4. Let $\mathcal{A}$ be any wf. of $\mathcal{L}$. Then, for any interpretation I, either $\mathcal{A}$ or $(\sim \mathcal{A})$ is true in I.

1. FALSE. That v and v' are i-equivalent means that $v(x_j) = v'(x_j), j \neq i$, not that $v(x_i) = v'(x_i)$. So, for instance, if $\mathcal{A}$ is the atomic formula $A_1^1(x_i)$ it is clear that v may satisfy $\mathcal{A}$ without v' doing so, and vice versa.

2. FALSE. A counterexample is $(A_1^1(x_1) \rightarrow A_1^1(x_1))$. This wf. is a substitution-instance of a tautology and hence, logically valid by Proposition 3.31 and Definition 3.35. However, the wf. is obviously not closed.

3. TRUE. If any valuation v in any interpretation I satisfies $\mathcal{A}$, then $(\sim \mathcal{A})$ is not satisfied by any v in any I. Hence, $(\sim \mathcal{A})$ is false in any I, and by Definition 3.35, $(\sim \mathcal{A})$ is contradictory.

4. FALSE. Let $\mathcal{A}$ be the wf. $A_1^1(x_1)$, let I be the interpretation with the integers $\mathbb{Z}$, and let the interpretation of A_1^1 be the predicate '> 0'. Clearly, there are valuations v and w, such that $v(x_1) > 0$ and $w(x_1) \leq 0$. Now, v satisfies $\mathcal{A}$ but not $(\sim \mathcal{A})$, while w satisfies $(\sim \mathcal{A})$ but not $\mathcal{A}$. Hence, neither $\mathcal{A}$ nor $(\sim \mathcal{A})$ is true in I.