SIF8015 Logic

Exercise 2, Solutions
Formal Statement Calculus

Caveat: In proofs and deductions, several parenthesis have been omitted, so in the strict sense of Definition 2.1 some of the expressions are not wfs. of L. For instance, parenthesis around "main" wfs. and negations are usually omitted.

Task 1

Write out proofs in L for the following wfs. Note that you are not allowed to use anything but axioms and Modus Ponens, in particular, you are not supposed to use neither the Deduction Theorem nor the rule Hypothetical Syllogism.

(1a) $(p_1 \to p_2) \to ((\sim p_1 \to \sim p_2) \to (p_2 \to p_1))$

$$
\begin{array}{lll}
(1) & (\sim p_1 \to \sim p_2) \to (p_2 \to p_1) & L3 \\
(2) & ((\sim p_1 \to \sim p_2) \to (p_2 \to p_1)) \to & \\
 & \quad ((p_1 \to p_2) \to ((\sim p_1 \to \sim p_2) \to (p_2 \to p_1))) & L1 \\
(3) & (p_1 \to p_2) \to ((\sim p_1 \to \sim p_2) \to (p_2 \to p_1)) & \text{MP}, (1), (2)
\end{array}
$$

(1b) $(\sim p_1 \to \sim p_2) \to (p_3 \to (p_2 \to p_1))$

$$
\begin{array}{lll}
(1) & (p_2 \to p_1) \to (p_3 \to (p_2 \to p_1)) & L1 \\
(2) & ((p_2 \to p_1) \to (p_3 \to (p_2 \to p_1))) \to & \\
 & \quad ((\sim p_1 \to \sim p_2) \to ((p_2 \to p_1) \to (p_3 \to (p_2 \to p_1)))) & L1 \\
(3) & (\sim p_1 \to \sim p_2) \to ((p_2 \to p_1) \to (p_3 \to (p_2 \to p_1))) & \text{MP}, (1), (2) \\
(4) & ((\sim p_1 \to \sim p_2) \to ((p_2 \to p_1) \to (p_3 \to (p_2 \to p_1)))) \to & \\
 & \quad (((\sim p_1 \to \sim p_2) \to (p_2 \to p_1)) \to & \\
 & \quad\quad (((\sim p_1 \to \sim p_2) \to (p_3 \to (p_2 \to p_1)))) & L2 \\
(5) & (((\sim p_1 \to \sim p_2) \to (p_2 \to p_1)) \to & \\
 & \quad (((\sim p_1 \to \sim p_2) \to (p_3 \to (p_2 \to p_1)))) & \text{MP}, (3), (4) \\
(6) & (\sim p_1 \to \sim p_2) \to (p_2 \to p_1) & L3 \\
(7) & (\sim p_1 \to \sim p_2) \to (p_3 \to (p_2 \to p_1)) & \text{MP}, (5), (6)
\end{array}
$$

Task 2

Do exercises 2a and 2c on page 36. Note that you are allowed to use results proved in examples, or other exercises in the textbook; however, you must write clearly which and why this gives your answer. You are not allowed to use truth tables.

(2a) Show that $\{(\sim \mathcal{A})\} \vdash_L (\mathcal{A} \to \mathcal{B})$

(1)	$\sim \mathcal{A}$	Assumption
(2)	$\sim \mathcal{A} \to (\sim \mathcal{B} \to\sim \mathcal{A})$	$L1$
(3)	$\sim \mathcal{B} \to\sim \mathcal{A}$	MP, (1), (2)
(4)	$(\sim \mathcal{B} \to\sim \mathcal{A}) \to (\mathcal{A} \to \mathcal{B})$	$L3$
(5)	$\mathcal{A} \to \mathcal{B}$	MP, (3), (4)

Hence, $\{(\sim \mathcal{A})\} \vdash_L (\mathcal{A} \to \mathcal{B})$.

(2c) Show that $\{(\mathcal{A} \to \mathcal{B}), (\sim (\mathcal{B} \to \mathcal{C}) \to (\sim \mathcal{A}))\} \vdash_L (\mathcal{A} \to \mathcal{C})$

(1)	$\mathcal{A} \to \mathcal{B}$	Assumption
(2)	$\sim (\mathcal{B} \to \mathcal{C}) \to\sim \mathcal{A}$	Assumption
(3)	$(\sim (\mathcal{B} \to \mathcal{C}) \to\sim \mathcal{A}) \to (\mathcal{A} \to (\mathcal{B} \to \mathcal{C}))$	$L3$
(4)	$\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$	MP, (2), (3)
(5)	$(\mathcal{A} \to (\mathcal{B} \to \mathcal{C})) \to ((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C}))$	$L2$
(6)	$(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})$	MP, (4), (5)
(7)	$\mathcal{A} \to \mathcal{C}$	MP, (1), (6)

Hence, $\{(\mathcal{A} \to \mathcal{B}), (\sim (\mathcal{B} \to \mathcal{C}) \to (\sim \mathcal{A}))\} \vdash_L (\mathcal{A} \to \mathcal{C})$.

Task 3

Show that, for any wfs. $\mathcal{A}$, and $\mathcal{B}$ of L, the following wfs. are theorems of L. You may use the Deduction Theorem, the rule Hypothetical Syllogism, and any other proven result from the textbook. You are not allowed to use truth tables.

(3a) $(\mathcal{A} \to (\sim (\sim \mathcal{A})))$

Note that in Exercise 2b in Hamilton, p. 36, it was proved that $\{\sim (\sim \mathcal{A})\} \vdash_L \mathcal{A}$, so by the Deduction Theorem, $(\sim\sim \mathcal{A} \to \mathcal{A})$ is a theorem of L. This holds for any wf. $\mathcal{A}$, in particular also for the wf. $(\sim \mathcal{A})$.

(1)	$\sim\sim\sim \mathcal{A} \to\sim \mathcal{A}$	Theorem, by above
(2)	$(\sim\sim\sim \mathcal{A} \to\sim \mathcal{A}) \to (\mathcal{A} \to\sim\sim \mathcal{A})$	$L3$
(3)	$\mathcal{A} \to\sim\sim \mathcal{A}$	MP, (1), (2)

(3b) $(\mathcal{A} \to (\sim \mathcal{B} \to (\sim (\mathcal{A} \to \mathcal{B}))))$

Note that in Exercise 3b in Hamilton, p. 36, it was proved that $((\mathcal{B} \to \mathcal{A}) \to (\sim \mathcal{A} \to \sim \mathcal{B}))$ is a theorem of L. Now, we prove first that $(\mathcal{A} \to ((\mathcal{A} \to \mathcal{B}) \to \mathcal{B}))$ is a theorem of L.

(1)	$\mathcal{A}$	Assumption
(2)	$\mathcal{A} \to \mathcal{B}$	Assumption
(3)	$\mathcal{B}$	MP, (1), (2)

Hence, by the Deduction Theorem (twice), $(\mathcal{A} \to ((\mathcal{A} \to \mathcal{B}) \to \mathcal{B}))$ is a theorem of L. Now we can prove that $(\mathcal{A} \to (\sim \mathcal{B} \to (\sim (\mathcal{A} \to \mathcal{B}))))$ as follows:

(4)	$\mathcal{A} \to ((\mathcal{A} \to \mathcal{B}) \to \mathcal{B})$	Above
(5)	$((\mathcal{A} \to \mathcal{B}) \to \mathcal{B}) \to (\sim \mathcal{B} \to \sim (\mathcal{A} \to \mathcal{B}))$	Ex. 3b, Ham.
(6)	$\mathcal{A} \to (\sim \mathcal{B} \to (\sim (\mathcal{A} \to \mathcal{B})))$	HS, (5), (6)

Task 4

Do exercise 5 on page 37.

Essentially, this is Example 2.6 on pp. 30-31, and the answer is **yes**, it is a legitimate deduction rule. The crux is to realize that it suffices to show that

$$\{\mathcal{B}, (\mathcal{A} \to (\mathcal{B} \to \mathcal{C}))\} \vdash_L (\mathcal{A} \to \mathcal{C})$$

i.e. that $(\mathcal{A} \to \mathcal{C})$ is deducible from $\Gamma = \{\mathcal{B}, (\mathcal{A} \to (\mathcal{B} \to \mathcal{C}))\}$.

Task 5

Do exercise 9 on page 44.

Prove that if $\mathcal{B}$ is a contradiction then $\mathcal{B}$ cannot be a theorem of any consistent extension of L.

Proof: Assume that $\mathcal{B}$ is a contradiction and that L^* is a consistent extension of L. Since $\mathcal{B}$ is a contradiction, $\mathcal{B}$ will have the truth value 0 (false) in any valuation. However, by Proposition 2.22, since L^* is a consistent extension of L, there exists a valuation v in which, every theorem of L^* evaluates to 1 (true). This gives a contradiction since $\mathcal{B}$ obviously cannot be both true and false in the same valuation v.

Task 6

Which of the following statements are true, explain briefly why or why not.

1. Every theorem of L is a tautology.
2. Every tautology has a proof in L.
3. If the wf. $\mathcal{A}$ is deducible from the set of wfs. Γ in L, then $\mathcal{A}$ is a theorem of L.
4. If $\mathcal{A}$ is a wf. of L, $\Gamma = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n\}$ is a set of wfs. of L, and $\Gamma \vdash_L \mathcal{A}$, then $(\mathcal{A}_1 \to (\mathcal{A}_2 \to \ldots (\mathcal{A}_{n-1} \to (\mathcal{A}_n \to \mathcal{A})) \cdots))$ is a theorem of L.
5. There are wfs. $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of L such that
$$\{(\mathcal{A} \to \mathcal{B}), (\mathcal{B} \to \mathcal{C})\} \vdash_L (\mathcal{A} \to \mathcal{C})$$
is a theorem of L.
6. For any L^* that is an inconsistent extension of L, there exists a wf. $\mathcal{B}$ possibly depending on L^* such that $\mathcal{B}$ is not a theorem of L^*.

Answers:

1. Yes, by the Soundness Theorem (Proposition 2.14).
2. Yes, by the Adequacy Theorem, all tautologies are theorems of L. And by definition, all theorems have a proof in L.
3. No, not in general. Only if Γ does not contain any non-theorem wfs., i.e. if $\Gamma = \emptyset$ or all wfs. of Γ are theorems of L.
4. Yes, repeated application of the Deduction Theorem gives that $\vdash_L (\mathcal{A}_1 \to (\mathcal{A}_2 \to \ldots (\mathcal{A}_{n-1} \to (\mathcal{A}_n \to \mathcal{A})) \cdots))$ so the formula is deducible from the axioms only, hence a theorem.
5. No, the expression
$$\{(\mathcal{A} \to \mathcal{B}), (\mathcal{B} \to \mathcal{C})\} \vdash_L (\mathcal{A} \to \mathcal{C})$$
is not a wf. of L. However, it expresses the deduction rule HS, which is a legitimate, derived deduction rule of L.
6. No, in any inconsistent extension of L, any wf. is a theorem, cf. Proposition 2.18.

Task 7

Prove $\vdash_L ((\sim (\mathcal{A} \to \mathcal{A})) \to \mathcal{B})$ for any wfs. $\mathcal{A}, \mathcal{B}$. How do you interpret this, when you know that $(\mathcal{A} \to \mathcal{A})$ is a tautology for any wf. $\mathcal{A}$? What can you say about an extension of L in which $(\sim (\mathcal{A} \to \mathcal{A}))$ is a theorem?

Proof:

$\vdash_L (\mathcal{A} \to \mathcal{A})$ is proved on page 32 as Example 2.7. We repeat the proof in order to make our proof self-contained:

(1)	$\mathcal{A} \to ((\mathcal{A} \to \mathcal{A}) \to \mathcal{A})$	$L1$
(2)	$(\mathcal{A} \to ((\mathcal{A} \to \mathcal{A}) \to \mathcal{A})) \to ((\mathcal{A} \to (\mathcal{A} \to \mathcal{A})) \to (\mathcal{A} \to \mathcal{A}))$	$L2$
(3)	$((\mathcal{A} \to (\mathcal{A} \to \mathcal{A})) \to (\mathcal{A} \to \mathcal{A}))$	MP, (1), (2)
(4)	$\mathcal{A} \to (\mathcal{A} \to \mathcal{A})$	$L1$
(5)	$\mathcal{A} \to \mathcal{A}$	MP, (4), (3)

This gives $\vdash_L (\mathcal{A} \to \mathcal{A})$. That is, $(\mathcal{A} \to \mathcal{A})$ can be deduced in L from the empty set of assumptions. Therefore, we also have

$$\{\sim \mathcal{B}\} \vdash_L (\mathcal{A} \to \mathcal{A}) \qquad (*)$$

for any wf. $\mathcal{B}$, since adding an assumption does not make a theorem not deducible (this is called monotony).

$$\vdash_L (\mathcal{A} \to \sim\sim \mathcal{A}) \qquad (**)$$

was proved in (3a), this exercise set. Together (*) and (**) gives us what we need to show that

$$((\sim \mathcal{B}) \to (\sim\sim (\mathcal{A} \to \mathcal{A}))) \qquad (***)$$

is a theorem of L. First prove $\{\sim \mathcal{B}\} \vdash_L (\sim\sim (\mathcal{A} \to \mathcal{A}))$ as follows:

(1)	$\{\sim \mathcal{B}\} \vdash_L (\mathcal{A} \to \mathcal{A})$	$(*)$
(2)	$\{\sim \mathcal{B}\} \vdash_L ((\mathcal{A} \to \mathcal{A}) \to \sim\sim (\mathcal{A} \to \mathcal{A})$	$(**)$
(3)	$\{\sim \mathcal{B}\} \vdash_L \sim\sim (\mathcal{A} \to \mathcal{A})$	MP, (1), (2)

The Deduction Theorem gives $\vdash_L ((\sim \mathcal{B}) \to (\sim\sim (\mathcal{A} \to \mathcal{A})))$ and then:

(1)	$\vdash_L \sim \mathcal{B} \to \sim\sim (\mathcal{A} \to \mathcal{A})$	$(***)$
(2)	$\vdash_L (\sim \mathcal{B} \to \sim\sim (\mathcal{A} \to \mathcal{A}) \to (\sim (\mathcal{A} \to \mathcal{A}) \to \mathcal{B})$	$L3$
(3)	$\vdash_L \sim (\mathcal{A} \to \mathcal{A}) \to \mathcal{B}$	MP, (4), (5)

Since $(\mathcal{A} \to \mathcal{A})$ is a tautology for any wf. $\mathcal{A}$, $(\sim (\mathcal{A} \to \mathcal{A}))$ must be a contradiction, i.e., false for any valuation. The proof of $\vdash_L ((\sim (\mathcal{A} \to \mathcal{A})) \to \mathcal{B})$ illustrates that from a contradiction it is possible, in L, to deduce any wf. Finally, any extension containing $(\sim (\mathcal{A} \to \mathcal{A}))$ as a theorem has to be inconsistent, as it contains any wf. as a theorem.