SIF8015 Logic

Exercise 1, Solutions
Informal Statement Calculus

Task 1

Translating natural language sentences into logical expressions can be very tricky, since natural language sentences often are ambiguous, depend on the context they are in, and require some external knowledge to be interpreted. It is therefore not uncommon for several alternative twists to be equally acceptable (even though they might not be logically equivalent), depending on what you think is actually meant and on the "world" you intend to describe. Furthermore, several different choices of how to decompose a sentence into statement variables might be equally acceptable, depending on the purpose of the logical formalization of the sentence.

1. If n is an even integer and m is an odd integer, then n^2m^2 will either be positive or zero.

p	n is an even integer
q	m is an odd integer
r	n^2m^2 is negative

$$((p \wedge q) \rightarrow (\sim r))$$

2. Your sorting algorithm will run in $O(n \log n)$ time on an average problem of size n if you have implemented either quicksort or insertsort with a binary search.

p	Your sorting algorithm will run in $O(n \log n)$ time on an average problem of size n
q	You have implemented quicksort
r	You have implemented insertsort with a binary search

$$((q \vee r) \rightarrow p)$$

3. The ratio between a rational number x and an integer y is either a rational number or undefined.

p	x is a rational number
q	y is an integer
r	$\frac{x}{y}$ is a rational number
s	$\frac{x}{y}$ is undefined

$$((p \wedge q) \rightarrow ((r \wedge (\sim s)) \vee (((\sim r) \wedge s)))$$

4. Your new disk drive will never rotate with the speed of light, unless you are Superman and there is no green cryptonite in the immediate vicinity, in which case it will when you spin it around by hand.

p	Your new disk drive rotates with the speed of light
q	You are Superman
r	There is no green cryptonite in the immediate vicinity
s	You spin the new disk drive around by hand

$$(((q \wedge r \wedge s) \rightarrow p) \wedge ((\sim (q \wedge r \wedge s)) \rightarrow (\sim p)))$$

5. "This statement is false" is either a true or a false statement, or it is not a statement at all.

p	"This statement is false" is a statement
q	"This statement is false" is a true statement
r	"This statement is false" is a false statement

$$((p \rightarrow (((\sim q) \wedge r) \vee (q \wedge (\sim r)))) \vee (\sim p))$$

6. Not only will your computer run a billion times faster than before and never crash, but when you install Microsoft's new operating system, you will either get a free lunch or hell will freeze over.

p	Your computer runs a billion times faster than before
q	Your computer will never crash
r	You have installed Microsoft's new operating system
s	You will get a free lunch
t	Hell will freeze over

$$(r \rightarrow (p \wedge q \wedge (s \vee t)))$$

Task 2

A Norwegian children's song contains the statements "min hatt den har tre kanter" and "har den ei tre kanter, så er den ei min hatt". Translate into symbols these two compound statements. Are the statements equivalent? Show why or why not.

p	Hatten er min
q	Hatten har tre kanter

(i) Min hatt den har tre kanter

$$(p \to q)$$

(ii) Har den ei tre kanter, så er den ei min hatt

$$((\sim q) \to (\sim p))$$

Statement (ii) is the contrapositive of statement (i), and the statements are indeed equivalent. A simple truth table will reveal that.

Task 3

Do exercises 3c, 3g, 3d, 5b, 5d, 6c, 6d and 7 on page 10 in the textbook by Hamilton.

Let "1" denote truth, and let "0" denote falsity.

(3c) Truth table for statement form $(p \to (q \to r))$:

p	q	r	$(q \to r)$	statement
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

(3d) Truth table for statement form $((p \wedge q) \to r)$:

p	q	r	$(p \wedge q)$	statement
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

Since the last column in this table is identical to the last column in the table from (3c), the two statements are logically equivalent since their biconditional then defines a tautology.

(3g) Truth table for statement form $(((\sim p) \wedge q) \to ((\sim q) \wedge r))$:

p	q	r	$((\sim p) \wedge q)$	$((\sim q) \wedge r)$	statement
0	0	0	0	0	1
0	0	1	0	1	1
0	1	0	1	0	0
0	1	1	1	0	0
1	0	0	0	0	1
1	0	1	0	1	1
1	1	0	0	0	1
1	1	1	0	0	1

(5b) Truth table for statement form $((q \vee r) \to ((\sim r) \to q))$:

q	r	$(q \vee r)$	$((\sim r) \to q)$	statement
0	0	0	0	1
0	1	1	1	1
1	0	1	1	1
1	1	1	1	1

The statement is a tautology since the last column in the table is all 1s.

(5d) Truth table for statement form $((p \to (q \to r)) \to ((p \wedge (\sim q)) \vee r))$:

p	q	r	$(p \to (q \to r))$	$((p \wedge (\sim q)) \vee r)$	statement
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	1	1	1

Since the last column in the table contains one or more 0s, the statement is not a tautology.

(6c) Create the truth table for each of the two statement forms. Since both tables have identical last columns, we can conclude that the two statement forms are logically equivalent beacause the biconditional of the two statement forms then constitutes a tautology.

(6d) Same procedure as (6c).

(7) Create the truth table for the given statement form. The statement form is not a tautology since the last column in its truth table contains one or more 0s.

Let $\mathcal{A}$ and $\mathcal{B}$ denote any two tautologies. Then, $(((\sim \mathcal{A}) \to \mathcal{B}) \to (\mathcal{A} \to (\sim \mathcal{B})))$ is a contradiction.

Task 4

Do exercise 11a on page 15 in the textbook by Hamilton.

(11a) Truth-preserving transformations:

$$\begin{aligned}
&((\sim (p \vee (\sim q))) \to (q \to r)) \equiv \\
&((\sim (p \vee (\sim q))) \to ((\sim q) \vee r)) \equiv \\
&((\sim ((\sim q) \vee p)) \to ((\sim q) \vee r)) \equiv \\
&((\sim (q \to p)) \to ((\sim q) \vee r))
\end{aligned}$$

Task 5

Do exercise 13d on page 19 in the textbook by Hamilton.

(13d) To find the conjunctive normal form of our statement, we first construct the truth table for its negation. From this, we can easily find the disjunctive normal form of the negation of our statement, by observing for which rows in the table this negation is true. By negating this disjunctive normal form and applying DeMorgan's laws, we arrive at the conjunctive normal form of our statement.

Note that the negation of a statement in disjunctive (conjunctive) normal form is a statement in conjuctive (disjunctive) normal form. The procedure above also employs the fact that $(\sim (\sim p))$ and (p) are logically equivalent.

$$\begin{aligned}
(((p \to q) \to r) \to s) \quad \equiv \quad & (p \vee q \vee (\sim r) \vee s) \wedge \\
& (p \vee (\sim q) \vee (\sim r) \vee s) \wedge \\
& ((\sim p) \vee q \vee r \vee s) \wedge \\
& ((\sim p) \vee q \vee (\sim r) \vee s) \wedge \\
& ((\sim p) \vee (\sim q) \vee (\sim r) \vee s)
\end{aligned}$$

Task 6

Do exercise 18 on page 22 in the textbook by Hamilton.

(18) The statement form $(p \to q)$ is logically equivalent to $((\sim p) \vee q)$. Note that $(\sim p)$ is logically equivalent to $(p|p)$, and that $(p \vee q)$ is logically equivalent to $((p|p)|(q|q))$. We thus make use of that logically equivalent expressions can be substituted for one another, without altering the truth value of the larger expression they occur in. Our original statement form is hence logically equivalent to $(((p|p)|(p|p))|(q|q))$, which in turn is logically equivalent to $(p|(q|q))$.

Task 7

Do exercise 21 on page 26 in the textbook by Hamilton.

(21) $((\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n) \to \mathcal{A})$ is by assumption a valid argument form, and is hence by definition then a tautology. Furthermore, $((\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n) \to \mathcal{A})$ is logically equivalent to $((\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_{n-1}) \to (\mathcal{A}_n \to \mathcal{A}))$, as shown below. $((\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_{n-1}) \to (\mathcal{A}_n \to \mathcal{A}))$ is hence also a tautology, and thus also a valid argument form.

$$
\begin{aligned}
&((\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n) \to \mathcal{A}) \equiv \\
&(\sim (\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n) \vee \mathcal{A}) \equiv \\
&(((\sim \mathcal{A}_1) \vee \ldots \vee (\sim \mathcal{A}_n)) \vee \mathcal{A}) \equiv \\
&((\sim \mathcal{A}_1) \vee \ldots \vee (\sim \mathcal{A}_{n-1})) \vee ((\sim \mathcal{A}_n) \vee \mathcal{A}) \equiv \\
&(\sim (\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_{n-1}) \vee ((\sim \mathcal{A}_n) \vee \mathcal{A})) \equiv \\
&((\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_{n-1}) \to ((\sim \mathcal{A}_n) \vee \mathcal{A})) \equiv \\
&((\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_{n-1}) \to (\mathcal{A}_n \to \mathcal{A}))
\end{aligned}
$$

A reductio ad absurdum proof is also possible.